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Representations of affine Lie superalgebras and their quantization in type A

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ABSTRACT

We construct a new family of irreducible modules over any basic classical affine Kac-Moody Lie superalgebra which are induced from modules over the Heisenberg subalgebra. We also obtain irreducible deformations of these modules for the quantum affine superalgebra $U_q \widehat{\mathfrak{sl}}(m|n)$.

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1. Introduction

Affine Lie algebras and superalgebras are among the most important algebraic structures with ample applications in modern sciences. While representations of affine Lie algebras were extensively studied and essentially understood, the representation theory of affine Lie superalgebras is largely undeveloped. The paper is a part of our program

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of studying the induced representations for affine Lie superalgebras and their quantizations. This study was initiated in [2] where the theory of Verma type modules for affine Lie superalgebras was developed. In the current paper, we consider parabolic induction in the case when the Levi subalgebra is an abelian extension of the Heisenberg subalgebra over the field of fractions $\mathbb{C}(q)$ and the inducing modules are diagonal modules. These modules have infinite dimensional weight subspaces and nonzero central charge. They also admit a natural eigenbasis for a certain infinite family of operators. Our first result establishes the irreducibility criterion of generalized imaginary Verma modules $M(\lambda, V) = \text{Ind}(\mathcal{P}, U(\widehat{\mathfrak{g}}); V)$ for affine Lie superalgebra $\widehat{\mathfrak{g}}$ associated with a parabolic subalgebra \mathcal{P} , whose Levi factor is a sum of the Heisenberg subalgebra and a Cartan subalgebra, an irreducible diagonal module V over the Heisenberg subalgebra and a weight λ .

Theorem 1.1. *For any irreducible diagonal module V over the Heisenberg subalgebra with nonzero central charge a and any weight λ such that $\lambda(c) = a$, the generalized imaginary Verma module $M(\lambda, V)$ is irreducible.*

This theorem is an analog of the corresponding result in [1] in the case of affine Lie algebras (see also [4], [8], [9], [10], [11]). Furthermore, differently from the other cases in the literature, we point out that we work with a specific order on the monomials of $U(\widehat{\mathfrak{g}})$. Such order is crucial in the study of quantum deformations of generalized imaginary Verma modules. Notice also that the category of diagonal modules over the Heisenberg subalgebra with a fixed nonzero central charge is equivalent to the category of weight modules for some Weyl algebra of infinite rank, whose irreducible objects were classified in [5]. Thus Theorem 1.1 gives a way to construct a large family of irreducible modules for affine Lie superalgebras based on these Weyl algebra representations. Namely, it defines a family of functors (parametrized by λ) from the category of weight modules over certain Weyl algebra of infinite rank to the category of modules over an affine Lie superalgebra each of which preserves irreducibility as long as $\lambda(c) \neq 0$.

For a distinguished parabolic subalgebra \mathcal{P}_Σ of $\widehat{\mathfrak{g}}$, the corresponding induced $\widehat{\mathfrak{g}}$ -module $M_\Sigma(\lambda, V)$ is isomorphic to a Kac module $K(M_0(\lambda, V))$, where \mathcal{P}_0 is the intersection of \mathcal{P}_Σ with the even part $\widehat{\mathfrak{g}}_0$ of $\widehat{\mathfrak{g}}$ and $M_0(\lambda, V) = \text{Ind}(\mathcal{P}_0, U(\widehat{\mathfrak{g}}_0); V)$ (see Theorem 3.4). Hence, in particular, Theorem 1.1 gives the irreducibility of the Kac module $K(M_0(\lambda, V))$.

Next, we pass to quantum affine superalgebras of type A . The restriction of the type is due to the fact that the PBW bases are only known for quantum algebras in type A [13], which we heavily depend on. We introduce the quantum analogs $M_q(\lambda, V_q) := \text{Ind}(\mathcal{P}_q, U_q(\widehat{\mathfrak{g}}); V_q)$ of generalized imaginary Verma modules, where V_q is an irreducible module over the quantum Heisenberg \mathcal{H}_q . We show that these modules are true quantum deformations of the corresponding classical modules using the technique of \mathbb{A} -forms [12]. In the non-super case, this approach has been successfully used in [6], where it was based on Beck's basis, and in [7].

Our second result is the following theorem which gives the irreducibility criterion for the quantum generalized imaginary Verma modules induced from generic V_q (cf. Corollary 4.14).

Theorem 1.2. *Let V_q be limit faithful (see Definition 4.10), and let λ be an integral weight with $\lambda(c) \neq 0$. Then the quantum generalized imaginary Verma $U_q \widehat{\mathfrak{sl}}(m|n)$ -module $M_q(\lambda, V_q)$ is irreducible if and only if V_q is irreducible.*

Since irreducible modules over the quantum Heisenberg subalgebra correspond to irreducible modules over the classical Heisenberg subalgebra (see Remark 2.1), Theorem 1.2 allows us to construct a family of irreducible modules for quantum affine Lie superalgebras of type A from weight irreducible modules over certain Weyl algebra of infinite rank.

The developed technique can be extended to more general parabolic subalgebras containing the Heisenberg subalgebra, [11]. We are going to address this in a forthcoming paper.

2. Preliminaries

Notation. Throughout the text q will denote an indeterminate and $\mathbb{C}(q)$ is the field of fractions of the polynomial ring $\mathbb{C}[q]$. We will use the subscript q to indicate that a vector space is defined over $\mathbb{C}(q)$. The elements of an algebra over $\mathbb{C}(q)$ (respectively, \mathbb{C}) will be represented by upper case (respectively, lower case) letters. If X, Y are sets, we denote by X^Y the set of all functions $f : Y \rightarrow X$.

2.1. Affine Kac-Moody Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ denote a basic classical Lie superalgebra of type $\mathfrak{gl}(m, n)$, $\mathfrak{sl}(m, n)$, $\mathfrak{psl}(n, n)$, $\mathfrak{osp}(m, n)$, $F(4)$, $G(2)$ or $D(2, 1; a)$ for $a \neq 0, -1$. Let $\mathfrak{h} \subset \mathfrak{g}_0$ be a fixed Cartan subalgebra of \mathfrak{g} and consider the root space decomposition of \mathfrak{g}

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$ and $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$. The set Δ is the root system of \mathfrak{g} , and \mathfrak{g}_α is the root space associated to the root α . Recall that every root is either purely even or purely odd, meaning that, for every $\alpha \in \Delta$, we have $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$ or $\mathfrak{g}_\alpha \subset \mathfrak{g}_1$, respectively. In particular, $\Delta = \Delta_0 \cup \Delta_1$, where $\Delta_i = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_i\}$ for $i \in \mathbb{Z}_2$. It is known that \mathfrak{g} is endowed with an even supersymmetric invariant bilinear form $(\cdot | \cdot)$, and that such a form is nondegenerate if \mathfrak{g} is not isomorphic to $\mathfrak{sl}(n, n)$.

Consider a set of simple roots $\Sigma = \{\alpha_i \mid i \in I\} \subset \Delta$ and the triangular decomposition $\Delta = \Delta^- \cup \Delta^+$ corresponding to Σ . Let $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$ and h_α for $\alpha \in \Delta^+$ denote the Chevalley type generators of \mathfrak{g} associated to Σ (in particular, $[x_\alpha^+, x_\alpha^-] = h_\alpha$ for all $\alpha \in \Delta^+$, and, if

α is even, then x_α^\pm, h_α is an \mathfrak{sl}_2 -triple). In what follows we set $x_i^\pm := x_{\alpha_i}^\pm$ and $h_i := h_{\alpha_i}$ for all $i \in I$. The triangular decomposition $\Delta = \Delta^- \cup \Delta^+$ induces a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where \mathfrak{n}^\pm is the subalgebra of \mathfrak{g} generated by x_i^\pm for $i \in I$. We point out that if \mathfrak{g} is isomorphic to $\mathfrak{psl}(n, n)$, then the set $\{h_i \mid i \in I\}$ is not linearly independent; in fact, in this case any h_i can be expressed as a linear combination of the remainder elements. Finally, let $A = (A_{i,j})$ be the Cartan matrix associated to Σ .

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the affine Kac-Moody Lie superalgebra corresponding to \mathfrak{g} . An element $z \otimes t^m \in \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ will be denoted by z_m (in particular, $h_{i,m} = h_i \otimes t^m$, $x_{i,m}^\pm = x_i^\pm \otimes t^m$ and $x_{\alpha,m}^\pm = x_\alpha^\pm \otimes t^m$). Moreover, for all $x, y, \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$ we have

$$[x_m, y_n] = [x, y]_{m+n} + m\delta_{m,-n}(x|y)c, \quad [c, \widehat{\mathfrak{g}}] = 0, \quad [d, x_m] = mx_m.$$

The Cartan subalgebra of $\widehat{\mathfrak{g}}$ is $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, and the root space decomposition of $\widehat{\mathfrak{g}}$ is given by

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \widehat{\Delta}} \widehat{\mathfrak{g}}_\alpha$$

where $\widehat{\Delta} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\} \subset \widehat{\mathfrak{h}}^*$ with $\delta \in \widehat{\mathfrak{h}}^*$ being defined so that $\delta(\mathfrak{h} + \mathbb{C}c) = 0$ and $\delta(d) = 1$. The sets $\widehat{\Delta}^{re} = \widehat{\Delta} \setminus \{n\delta \mid n \in \mathbb{Z}\}$ and $\widehat{\Delta}^{im} = \{n\delta \mid n \in \mathbb{Z}\}$ are called the sets of real and imaginary roots of $\widehat{\mathfrak{g}}$, respectively. It is easy to see that, for all $\alpha \in \Delta^+$ and $n \in \mathbb{Z} \setminus \{0\}$, we have $\widehat{\mathfrak{g}}_{\pm\alpha+n\delta} = \mathbb{C}x_{\alpha,n}^\pm$ and $\widehat{\mathfrak{g}}_{n\delta} = \mathfrak{h} \otimes t^n$. Recall also that the bilinear form $(\cdot|\cdot)$ of \mathfrak{g} can be extended to an even supersymmetric invariant bilinear form on $\widehat{\mathfrak{g}}$, which is nondegenerate if \mathfrak{g} is not isomorphic to $\mathfrak{sl}(n, n)$; namely we have

$$(x_m|y_n) = \delta_{m,-n}(x|y), \quad (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] | \mathbb{C}c \oplus \mathbb{C}d) = 0, \quad (c|c) = (d|d) = 0, \quad (c|d) = 1.$$

If \mathfrak{g} is not isomorphic to $\mathfrak{sl}(n, n)$, then the subalgebra

$$\mathcal{H}' = \mathbb{C}c \oplus \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \widehat{\mathfrak{g}}_{n\delta} \subset \widehat{\mathfrak{g}}$$

is isomorphic to an infinite-dimensional \mathbb{Z} -graded Heisenberg Lie algebra. If \mathfrak{g} is isomorphic to $\mathfrak{sl}(n, n)$, then \mathfrak{h} contains the identity matrix $z := I_{2n, 2n}$ of \mathfrak{g} , and the subalgebra \mathcal{H} is isomorphic to a direct sum of an infinite-dimensional \mathbb{Z} -graded Heisenberg Lie algebra and a commutative Lie algebra generated by the vectors $z_r, r \in \mathbb{Z}^\times$.

In what follows we will consider the algebra

$$\mathcal{H} = \mathcal{H}' \oplus \mathbb{C}d.$$

2.2. Quantum Heisenberg algebra

Consider the $\mathbb{C}(q)$ -vector space $\mathbb{C}(q) \otimes_{\mathbb{C}} \mathcal{H}'$, set $H := 1 \otimes h$ for any $h \in \mathfrak{h}$, $H_{i,r} := 1 \otimes h_{i,r}$ for all $i \in I$, and $q^c := 1 \otimes c$. Let $\widetilde{\mathcal{H}}_q$ be the free associative algebra over $\mathbb{C}(q)$ associated

to the vector space $\mathbb{C}(q) \otimes_{\mathbb{C}} \mathcal{H}'$. The quantum Heisenberg algebra \mathcal{H}'_q is defined to be $\widetilde{\mathcal{H}}_q$ modulo the following relations:

$$q^{\pm c} \text{ is central, } [H_{i,r}, H_{j,s}] = \delta_{r,-s} \frac{[rA_{i,j}][rc]}{r}$$

where $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ and $[kc] = \frac{q^{kc} - q^{-kc}}{q - q^{-1}}$ for all $k \in \mathbb{Z}$.

Remark 2.1. Note that \mathcal{H}'_q is not a true quantization of the algebra \mathcal{H}' . More precisely, for any $a \in \mathbb{Z}$ we have an isomorphism of algebras $\mathcal{H}'_q / \langle q^c - q^a \rangle \cong \mathbb{C}(q) \otimes_{\mathbb{C}} \mathcal{H}' / \langle c - a \rangle$.

In what follows we consider the algebra

$$\mathcal{H}_q = \mathcal{H}'_q \oplus \mathbb{C}q^d \quad \text{where} \quad q^d H_{i,r} q^{-d} = q^r H_{i,r} \text{ for all } i \in I, r \in \mathbb{Z}^\times.$$

2.3. Diagonal modules over quantum Heisenberg algebras

Assume that \mathfrak{g} is not isomorphic to $\mathfrak{psl}(n, n)$ or $\mathfrak{sl}(n, n)$. In these cases $\det A \neq 0$. For each $r \in \mathbb{Z}^\times$, define the matrix $A(r) := ([rA_{i,j}])_{i,j \in I}$. Since $\lim_{q \rightarrow 1} [rA_{i,j}] = rA_{i,j}$, we have $\det A(r) \neq 0$ for all $r \in \mathbb{Z}^\times$. Setting $\phi_{i,r} = H_{i,r}$ for $r > 0$, $i \in I$ and using the fact that $\det A \neq 0$, we can find $\phi_{i,r}$ for $r < 0$ such that $\{\phi_{i,k} \mid (i, k) \in I \times \mathbb{Z}^\times\}$ generates \mathcal{H}_q and

$$[\phi_{i,r}, \phi_{j,s}] = \delta_{i,j} \delta_{r,-s} [rc], \quad \text{for all } i, j \in I, r, s \in \mathbb{Z}^\times. \tag{2.1}$$

Since each $\phi_{i,r}$ is a linear combination of the vectors $H_{i,r}$, $i \in I$, we see that $q^d \phi_{i,r} q^{-d} = q^r \phi_{i,r}$ for all $i \in I, r \in \mathbb{Z}^\times$.

We say a \mathcal{H}_q -module V_q is graded if $V_q = \bigoplus_{r \in \mathbb{Z}} V_{q,r}$, where $V_{q,r}$ is the eigenspace of q^d associated with the eigenvalue q^r . A graded \mathcal{H}_q -module V_q is called *diagonal* if the operators $\phi_{i,k} \phi_{i,-k}$ have a common eigenvector (with eigenvalues living in $\mathbb{C}(q)$) in V_q for all $(i, k) \in I \times \mathbb{Z}_{>0}$. If V_q is a diagonal irreducible \mathcal{H}_q -module, then the elements $\phi_{i,k} \phi_{i,-k}$ are simultaneously diagonalizable on V_q for all $(i, k) \in I \times \mathbb{Z}_{>0}$. If v is such an eigenvector with $\phi_{i,k} \phi_{i,-k} v = \lambda_{ik} v$, then observe that

$$\phi_{i,k} \phi_{i,-k} \phi_{i,\pm k}^p v = (\lambda_{ik} \mp p[kc]) \phi_{i,\pm k}^p v, \quad \phi_{i,k} \phi_{i,-k} \phi_{i,r}^p v = \lambda_{ik} \phi_{i,r}^p v.$$

In what follows, we let $\mathcal{K}_{q,a}$ be the category of all diagonal graded \mathcal{H}_q -modules such that q^c acts as q^a with $a \in \mathbb{Z}^\times$.

Fix $a \in \mathbb{Z}^\times$ and $\mu \in \mathbb{C}(q)^{I \times \mathbb{Z}_{>0}}$ such that $\mu(i, k) = \mu_{i,k}$. Define the \mathcal{H}_q -module

$$V_q(\mu, a) = \mathcal{H}_q / \mathcal{J}_{\mu,a},$$

where $\mathcal{J}_{\mu,a}$ is the left ideal of \mathcal{H}_q generated by

$$\{\phi_{i,k}\phi_{i,-k} - \mu_{i,k}, q^c - q^a, q^d \mid (i, k) \in I \times \mathbb{Z}_{>0}\}.$$

Let $v_{\mu,a}$ denote the image of 1 in $V_q(\mu, a)$. In order to construct a basis for $V_q(\mu, a)$, we consider the set Φ of all functions $\varphi : I \times \mathbb{Z}_{>0} \rightarrow \{\pm 1\}$, and the set Π of all functions $\pi : I \times \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{\geq 0}$ with finite support. Given $(\varphi, \pi) \in \Phi \times \Pi$, define

$$\phi_{(\varphi,\pi)} = \prod_{(i,k) \in I \times \mathbb{Z}_{>0}} \phi_{i,\varphi(i,k)k}^{\pi(i,k)} \in \mathcal{H}_q.$$

Notice that all factors in $\phi_{(\varphi,\pi)}$ commute among themselves, so the product can be taken in any order.

Lemma 2.2. *The following statements hold:*

- (i) *The set $\{\phi_{(\varphi,\pi)}v_{\mu,a} \mid (\varphi, \pi) \in \Phi \times \Pi\}$ is a basis of $V_q(\mu, a)$*
- (ii) *The module $V_q(\mu, a)$ is irreducible if and only if $\mu_{i,k} \notin [ka]\mathbb{Z}$ for all $(i, k) \in I \times \mathbb{Z}_{>0}$.*

Proof. (i): To see that this set generates $V_q(\mu, a)$ we just observe that any monomial that has $\phi_{i,k}$ and $\phi_{i,-k}$ as factors lies in $\mathcal{J}_{\mu,a}$. The linear independence follows from the fact that the $\phi_{i,k}\phi_{i,-k}$ -eigenvalues of each $\phi_{(\varphi,\pi)}v_{\mu,a}$ is uniquely determined by (φ, π) .

(ii): Notice that

$$\phi_{i,-k}\phi_{i,k}^n v_{\mu,a} = (-n[ka] + \mu_{i,k})\phi_{i,k}^{n-1} v_{\mu,a}, \text{ and } \phi_{i,k}\phi_{i,-k}^n v_{\mu,a} = ((n-1)[ka] + \mu_{i,k})\phi_{i,-k}^{n-1} v_{\mu,a}.$$

Hence, if $\mu_{i,k} = n[ka]$ for some $n \in \mathbb{Z}_{>0}$ (respectively, $n \in \mathbb{Z}_{\leq 0}$), then the vector $\phi_{i,k}^n v_{\mu,a}$ (respectively, $\phi_{i,-k}^{-n+1} v_{\mu,a}$) generates a proper submodule of $V_q(\mu, a)$. The converse is clear. \square

It follows from Lemma 2.2 that the module $V_q(\mu, a)$ is reducible if and only if $F_{\mu,a} = \{(i, k) \in I \times \mathbb{Z}_{>0} \mid \mu_{i,k} \in [ka]\mathbb{Z}\} \neq \emptyset$. For $F \subseteq F_{\mu,a}$, we define $V_q(\mu, a, F)$ to be the quotient of $V_q(\mu, a)$ by its submodule generated by $\{(\phi_{i,k})^{\frac{\mu_{i,k}}{[ka]}} v_{\mu,a} \text{ if } \frac{\mu_{i,k}}{[ka]} > 0, (\phi_{i,-k})^{1-\frac{\mu_{i,k}}{[ka]}} v_{\mu,a} \text{ if } \frac{\mu_{i,k}}{[ka]} \leq 0 \mid (i, k) \in F\}$. We continue to denote the image of $v_{\mu,a}$ in $V_q(\mu, a, F)$ by $v_{\mu,a}$.

To construct a basis for $V_q(\mu, a, F)$ we consider the sets

$$E_1 = \{(\phi_{i_1,k_1})^{j_1} \cdots (\phi_{i_l,k_l})^{j_l} v_{\mu,a} \mid (i_j, k_j) \in F, 0 \leq j_s < \frac{\mu_{i_s,k_s}}{[k_s a]}\},$$

$$E_2 = \{(\phi_{i_1,-k_1})^{j_1} \cdots (\phi_{i_l,-k_l})^{j_l} v_{\mu,a} \mid v_{\mu,a} \in E_1, (i_j, k_j) \in F, 0 \leq j_s \leq -\frac{\mu_{i_s,k_s}}{[k_s a]}\}.$$

Next, we let $\Phi(F) \subseteq \Phi$ denote the set of all functions $\varphi \in \Phi$ for which the following hold

$$\varphi(i, k) = \begin{cases} -1, & \text{if } \frac{\mu_{i,k}}{[ka]} > 0; \\ 1, & \text{if } \frac{\mu_{i,k}}{[ka]} \leq 0, \end{cases}$$

for all $(i, k) \in F$. For all $v \in E_2$ we define $\Phi(v) = \{\varphi \in \Phi(F) \mid \phi_{i,-\varphi(i,k)k}$ is not a factor occurring in $v\}$, then

$$\{\phi_{(\varphi,\pi)v} \mid v \in E_2, (\varphi, \pi) \in \Phi(v) \times \Pi\}$$

is a basis of $V_q(\boldsymbol{\mu}, a, F)$.

Lemma 2.3. *For any fixed $j \in I$, the vectors $H_{j,k}v_{\boldsymbol{\mu},a}$ and $H_{j,-k}v_{\boldsymbol{\mu},a}$ cannot vanish simultaneously.*

Proof. If $H_{j,k}v_{\boldsymbol{\mu},a} = 0$, then $\phi_{j,k}v_{\boldsymbol{\mu},a} = 0$. Moreover, if $H_{j,-k}v_{\boldsymbol{\mu},a} = 0$, then $\sum_{i \in I} b_{i,k}\phi_{i,-k}v_{\boldsymbol{\mu},a} = 0$. This together with the linearly independence of the vectors $\phi_{i,-k}v_{\boldsymbol{\mu},a}$ (they have different $\phi_{i,k}\phi_{i,-k}$ -eigenvalues) imply that $\phi_{i,-k}v_{\boldsymbol{\mu},a} = 0$ for all $i \in I$. But then we have $\phi_{j,\pm k}v_{\boldsymbol{\mu},a} = 0$, contradicting the fact that $\phi_{i,\varphi(i,k)k}v_{\boldsymbol{\mu},a} \neq 0$ for any $i \in I, k \in \mathbb{Z}^\times$. \square

It is worth to notice that Lemma 2.3 in the non-super setting is a consequence of the fact that $A_{j,j} \neq 0$ for all $j \in I$. Such argument clearly does not work for isotropic roots.

Finally, we define $\Phi_{\boldsymbol{\mu},a} = \Phi(F_{\boldsymbol{\mu},a})$, and we write $\varphi = \varphi_{\boldsymbol{\mu},a}$ to indicate that a function φ lies in $\Phi_{\boldsymbol{\mu},a}$. Note that $V_q(\boldsymbol{\mu}, a, F_{\boldsymbol{\mu},a})$ is irreducible and that $V_q(\boldsymbol{\mu}, a, \emptyset) = V_q(\boldsymbol{\mu}, a)$. It is obvious that the modules of the form $V_q(\boldsymbol{\mu}, a, F)$ and $V_q(\boldsymbol{\nu}, a, F')$ are not isomorphic if $F_{\boldsymbol{\mu},a} \neq F_{\boldsymbol{\nu},a}$ or $F \neq F'$. Moreover we have the following result:

Proposition 2.4. *Let $F \subseteq F_{\boldsymbol{\mu},a} = F_{\boldsymbol{\nu},a}$. Then the modules $V_q(\boldsymbol{\mu}, a, F)$ and $V_q(\boldsymbol{\nu}, a, F)$ are isomorphic if and only if $\mu_{i,k} = \nu_{i,k}$ for all but finitely many indices, $\mu_{i,k} - \nu_{i,k} \in [ka]\mathbb{Z}$ for all $(i, k) \in I \times \mathbb{Z}_{>0}$, and $\varphi_{\boldsymbol{\mu},a}(i, k) = \varphi_{\boldsymbol{\nu},a}(i, k)$ for all $(i, k) \in F_{\boldsymbol{\mu},a}$.*

Proof. Fix $(j, \ell) \in I \times \mathbb{Z}_{>0}$ and let $\boldsymbol{\kappa} = (\kappa_{i,k}) \in \mathbb{C}(q)^{I \times \mathbb{Z}_{>0}}$ be defined as $\kappa_{j,\ell} = \mu_{j,\ell} + [\ell a]$ and $\kappa_{i,k} = \mu_{i,k}$ if $(i, k) \neq (j, \ell)$. Then, we have a homomorphism of \mathcal{H}_q -modules $\xi_{j,\ell} : V_q(\boldsymbol{\mu}, a) \rightarrow V_q(\boldsymbol{\kappa}, a)$ which maps $v_{\boldsymbol{\mu},a}$ to $\phi_{j,\ell}v_{\boldsymbol{\kappa},a}$. Moreover, $\xi_{j,\ell}$ is an isomorphism if and only if $\mu_{j,\ell} \neq 0$ which is equivalent to ask that $\varphi_{\boldsymbol{\mu},a}(i, k) = \varphi_{\boldsymbol{\kappa},a}(i, k)$ for all $(i, k) \in F_{\boldsymbol{\mu},a}$. Observe that, in this case the isomorphism $\xi_{j,\ell}^{-1}$ maps $v_{\boldsymbol{\kappa},a}$ to $\phi_{j,-\ell}v_{\boldsymbol{\mu},a}$. Furthermore, the isomorphism $\xi_{j,\ell}$ maps the kernel of the canonical map $V_q(\boldsymbol{\mu}, a) \rightarrow V_q(\boldsymbol{\mu}, a, F)$ to the kernel of the canonical map $V_q(\boldsymbol{\kappa}, a) \rightarrow V_q(\boldsymbol{\kappa}, a, F)$ for any $F \subseteq F_{\boldsymbol{\mu},a}$. Hence, it induces an isomorphism between $V_q(\boldsymbol{\mu}, a, F)$ and $V_q(\boldsymbol{\kappa}, a, F)$.

Finally, if $V_q(\boldsymbol{\mu}, a, F)$ and $V_q(\boldsymbol{\nu}, a, F)$ satisfy the hypothesis of the proposition, then the desirable isomorphism between such modules is given by composition of finitely many suitable powers of $\xi_{j,\ell}$, or its inverse, for all $(j, \ell) \in I \times \mathbb{Z}_{>0}$ such that $\mu_{j,\ell} \neq \nu_{j,\ell}$. \square

2.4. Some remarks

Each $\varphi \in \Phi$ induces a triangular decomposition of \mathcal{H}_q

$$\mathcal{H}_q = \mathcal{H}_q^{-\varphi} \otimes \mathbb{C}(q)[q^{\pm c}] \otimes \mathcal{H}_q^{\varphi},$$

where $\mathcal{H}_q^{\pm\varphi}$ is the subalgebra of \mathcal{H}_q generated by $\phi_{i,\pm\varphi(i,r)r}$, $(i, r) \in I \times \mathbb{Z}_{>0}$. The φ -Verma module of level a is the \mathcal{H}_q -module $V^\varphi(a)$ generated by a nonzero vector $v_{\varphi,a}$ such that $q^c v_{\varphi,a} = q^a v_{\varphi,a}$ and $\phi_{i,\varphi(i,r)r} v_{\varphi,a} = 0$ for all $(i, r) \in I \times \mathbb{Z}_{>0}$. As a vector space, $V^\varphi(a)$ is isomorphic to $\mathcal{H}_q^- \varphi$. We point out that when $\frac{\mu_{i,k}}{[ka]} \in \{0, 1\}$ for all $(i, k) \in I \times \mathbb{Z}_{>0}$, then $\Phi_{\mu,a} = \{\psi\}$ and $V_q(\mu, a, F_{\mu,a})$ is isomorphic to the φ -Verma module $V^\varphi(a)$ with $\varphi = -\psi$.

The following subalgebra of $\mathbb{C}(q)$ will be important in the subsequent sections:

$$\mathbb{A} = \left\{ \frac{f(q)}{g(q)} \in \mathbb{C}(q) \mid g(1) \neq 0 \right\} \subseteq \mathbb{C}(q).$$

Remark 2.5.

- (i) The construction of classical analogs of modules of the form $V_q(\mu, a, F)$ also works for the classical Heisenberg algebra \mathcal{H} for all $a \in \mathbb{C}^\times$. The only difference is that the eigenvalues $\mu_{i,k}$ should live in \mathbb{C} instead of $\mathbb{C}(q)$. Any $\mu \in \mathbb{A}^{I \times \mathbb{Z}_{>0}}$ induces $\mu(1) \in \mathbb{C}^{I \times \mathbb{Z}_{>0}}$ where $\mu(1)(i, k) = \mu_{i,k}(1)$, and we denote the \mathcal{H} -module $V(\mu(1), a, F)$ also by $V(\mu, a, F)$ (we just drop the q from the notation). We denote by $L(\mu, a)$ the irreducible quotient of $V(\mu, a)$. We notice however that $V(\mu, a, F_{\mu,a})$ is reducible in general (in fact, if some $\mu_{i,k}(1) \in a\mathbb{Z}$, but $\mu_{i,k} \notin [ka]\mathbb{Z}$, then $V(\mu, a, F_{\mu,a})$ is not an irreducible \mathcal{H} -module) and $L(\mu, a)$ is its irreducible quotient.
- (ii) The category \mathcal{K}_a of diagonal \mathcal{H} -modules of level a was studied in [1,5]. A non-trivial irreducible \mathcal{H} -module in \mathcal{K}_a has finite-dimensional graded subspaces if and only if it is isomorphic to a φ -Verma module with φ constant [1]. Moreover, it was proved in [5] that any simple module in \mathcal{K}_a is isomorphic to some $L(\mu, a)$.

Remark 2.6.

- (i) If $\mathfrak{g} = \mathfrak{psl}(n, n)$, then the elements $h_i \in \mathfrak{h}$ with $i \in I$ are no longer linearly independent. However, each h_j is a linear combination of h_i for $i \in I \setminus \{j\}$, and hence, the action of $H_{|\Sigma|,r}$ on any \mathcal{H}_q -module is determined by the action of the other generators $H_{i,r}$ for $i \in I \setminus \{|\Sigma|\}$. In this case, we consider $I' = \{1, \dots, |\Sigma| - 1\}$ and we get a linearly independent set $\{h_i \mid i \in I'\}$ which is a basis of \mathfrak{h} . Now all constructions of this subsection will follow by replacing I by I' everywhere and A by the matrix A' obtained by deleting the latter row and column of A . We also point out that Lemma 2.3 still holds for all $j \in I$ in this case (just replace I by I' in the proof).
- (ii) If $\mathfrak{g} = \mathfrak{sl}(n, n)$, then there is a linear combination of the vectors h_i , $i \in I$ resulting in the identity matrix $z := I_{2n,2n} \in \mathfrak{h}$. Then, for all $r \in \mathbb{Z}^\times$, the vector $Z_r = 1 \otimes z_r$ is central in \mathcal{H}'_q , and it must act trivially on any simple graded \mathcal{H}_q -module. In other words, any simple \mathcal{H}_q -module is just the pull back (under the map induced by the canonical projection $\mathfrak{sl}(n, n) \rightarrow \mathfrak{psl}(n, n)$) of a simple module over the \mathfrak{psl} -type Heisenberg algebra. Motivated by this, the \mathcal{H}_q -modules we will consider in

this paper are precisely those obtained via such pull back from diagonal modules over the \mathfrak{psl} -type Heisenberg algebra. This implies that the vectors $Z_r = 1 \otimes z_r$ act trivially on any such module, and thus the action of $H_{|\Sigma|,r}$ is determined by the action of the other generators $H_{i,r}, i \in I'$. Hence, as in the \mathfrak{psl} case, all constructions of this subsection will follow by replacing I by I' and A by A' everywhere. Also similarly to the \mathfrak{psl} case, we have that Lemma 2.3 still holds for all $j \in I$.

3. Induced $\widehat{\mathfrak{g}}$ -modules

In this section we consider modules over $\widehat{\mathfrak{g}}$ which are induced from modules over \mathcal{H} . For this we take a system of simple roots $\Sigma = \{\alpha_1, \dots, \alpha_r\}$ of Δ , and we recall that for any subset $X \subset \Sigma$ we can define $\Delta(X) = \mathbb{Z}X \cap \Delta$ and $\Delta(X)^+ = \Delta(X) \cap \Delta^+$. Then the set

$$\begin{aligned} \widehat{\Delta} \supseteq P(\Sigma, X) &= \{\alpha + n\delta \mid \alpha \in \Delta^+ \setminus \Delta(X)^+, n \in \mathbb{Z}\} \\ &\cup \{\alpha + n\delta \mid \alpha \in \Delta(X) \cup \{0\}, n \in \mathbb{Z}_{>0}\} \cup \Delta(X)^+ \end{aligned}$$

is a parabolic partition of $\widehat{\Delta}$. Moreover, any parabolic partition P of $\widehat{\Delta}$ is of this form for some choice of Σ and of $X \subseteq \Sigma$. Borel subalgebras of $\widehat{\mathfrak{g}}$ are in bijection with parabolic partitions of $\widehat{\Delta}$; namely, for any given parabolic partition P we can define the Borel subalgebra $\mathcal{B} = \widehat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in P} \widehat{\mathfrak{g}}_\alpha$.

In what follows we will be mainly interested in the so called natural Borel subalgebras. These are given by choosing $X = \emptyset$. Notice that in this case $\mathcal{B} = \widehat{\mathfrak{h}} \oplus (\mathfrak{h} \otimes t\mathbb{C}[t]) \oplus (\mathfrak{n}^+ \otimes \mathbb{C}[t, t^{-1}])$. Now we define the following subalgebras of $\widehat{\mathfrak{g}}$:

$$\mathcal{U}^\pm = \mathfrak{n}^\pm \otimes \mathbb{C}[t, t^{-1}], \quad \mathcal{N}^\pm = (\mathfrak{h} \otimes t\mathbb{C}[t]) \oplus \mathcal{U}^\pm, \quad \mathcal{P} = \mathfrak{h} \oplus \mathcal{H} \oplus \mathcal{U}^+.$$

The subalgebra \mathcal{P} is a minimal parabolic subalgebra of $\widehat{\mathfrak{g}}$ containing $\mathcal{B} + \mathcal{H}$. The Heisenberg algebra \mathcal{H} is called the Levi factor of \mathcal{P} .

Let V be a \mathcal{H} -module in \mathcal{K}_a and let $\lambda \in \widehat{\mathfrak{h}}^*$ such that $\lambda(c) = a$. We extend V to a \mathcal{P} -module by setting $hv = \lambda(h)v$ for all $v \in V$ and $h \in \mathfrak{h}$, and by letting \mathcal{U}^+ act trivially on it. Consider the induced $U(\widehat{\mathfrak{g}})$ -module

$$M(\lambda, V) = \text{Ind}(\mathcal{P}, U(\widehat{\mathfrak{g}}); V) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathcal{P})} V.$$

When V is simple, $M(\lambda, V)$ has a unique maximal module and hence a unique simple quotient. The $U(\widehat{\mathfrak{g}})$ -module $M(\lambda, V)$ is called the *generalized imaginary Verma module* associated with \mathcal{P}, V and λ . If V is a φ -Verma module of \mathcal{H} , then $M^\varphi(\lambda) := M(\lambda, V)$ is called the φ -*imaginary Verma module* of weight λ . If $\varphi \equiv 1$, we write $M(\lambda) = M^\varphi(\lambda)$, and we have the following result from [2].

Theorem 3.1 ([2]). *$M(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.*

In the next section we will generalize Theorem 3.1 to the case where V is an arbitrary irreducible module in \mathcal{K}_a .

3.1. Irreducibility criterion

Let $\mu \in \mathbb{C}^{I \times \mathbb{Z}_{>0}}$, $\lambda \in \mathfrak{h}^*$ with $\lambda(c) = a \neq 0$, and $F \subseteq F_{\mu,a}$. Take the \mathcal{H} -module $V = V(\mu, a, F)$ as defined in Subsection 2.3 and consider the associated generalized imaginary Verma $U(\widehat{\mathfrak{g}})$ -module $M(\lambda, V)$. For simplicity, write $v_j = \phi_{(\varphi,\pi)} v_{\mu,a}$ for $j = (\varphi, \pi) \in \Phi \times \Pi$, and denote by d_j the degree of the vector v_j .

Consider the total order “ \leq ” on Δ^+ defined as follows: for $i, j \in I$, set $\alpha_i \leq \alpha_j$ if $i \leq j$, then extend the ordering to Δ^+ by setting $\alpha \leq \beta \leq \gamma$ if $\beta = \alpha + \gamma$ and $\alpha \leq \gamma$. This induces a total order on $\Delta^+ \times \mathbb{Z}$ given by

$$(\alpha, r) \leq (\alpha', r') \text{ if and only if } \alpha < \alpha' \text{ or } \alpha = \alpha', r \leq r'. \tag{3.1}$$

Let M be the set of all functions $m : \Delta^+ \times \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ with finite support and such that $m(\alpha, k) \leq 1$ if $|\alpha| = 1$. Consider M equipped with the lexicographical order induced by the total order on $\Delta^+ \times \mathbb{Z}$; that is, given $m_1, m_2 \in M$, let $(\alpha_0, k_0) = \min\{(\alpha, k) \in \Delta^+ \times \mathbb{Z} \mid m_1(\alpha, k) \neq m_2(\alpha, k)\}$, then

$$m_1 < m_2, \text{ if } m_1(\alpha_0, k_0) < m_2(\alpha_0, k_0). \tag{3.2}$$

Now, this order on M induces in the obvious way a total order on the set of all monomials

$$x_m^+ := \prod_{(\alpha,r) \in \Delta^+ \times \mathbb{Z}}^{\rightarrow} (x_{\alpha,r})^{m(\alpha,r)}, \quad x_m^- := \prod_{(\alpha,r) \in \Delta^+ \times \mathbb{Z}}^{\leftarrow} (x_{-\alpha,r})^{m(\alpha,r)},$$

where the \rightarrow over the product indicates that the product is written in increasing order from left to right with respect to the order on $\Delta^+ \times \mathbb{Z}$, and the opposite for \leftarrow .

Since we have an isomorphism of $\widehat{\mathfrak{h}}$ -modules $M(\lambda, V) \cong U(\mathcal{U}^-) \otimes V$, the set of monomials

$$\{x_m^- v_j \mid m \in M, j \in \Phi(F) \times \Pi\}$$

is a basis of $M(\lambda, V)$ consisting of weight vectors. Define $Q = \mathbb{Z}\Sigma$ and $Q^+ = \mathbb{Z}_{\geq 0}\Sigma$, and let $v \in M(\lambda, V)_\mu$ be a nonzero vector of weight $\mu = \lambda - \beta + n\delta$, for some $\beta \in Q^+ \setminus \{0\}$, $n \in \mathbb{Z}$. Since $\beta \in Q^+ \setminus \{0\}$, there exist unique $a_i \in \mathbb{Z}_{\geq 0}$ such that $\beta = \sum_{i=1}^{N-1} a_i \alpha_i$. Define

$$\text{ht}(v) := \text{ht}(\beta) = \sum_{i=1}^{N-1} a_i.$$

Although there are some versions of the next result, none of them consider the order we fixed on the monomials of $U(\widehat{\mathfrak{g}})$. Such order will be crucial for the next section where we study quantum deformations of generalized imaginary Verma modules.

Lemma 3.2. *Let $\mu \in \mathbb{C}^{I \times \mathbb{Z}_{>0}}$, $\lambda \in \mathfrak{h}^*$ with $\lambda(c) = a \neq 0$, and $F \subseteq F_{\mu,a}$. Let V be an \mathcal{H} -module of the form $V(\mu, a, F)$ or $L(\mu, a)$. If $v \in M(\lambda, V)$ is a nonzero weight vector such that $\text{ht}(v) > 0$, then there exists $y \in U(\widehat{\mathfrak{g}})$ such that $yv \neq 0$ and $\text{ht}(yv) < \text{ht}(v)$.*

Proof. Write

$$v = \sum_{j \in L} a_j x_{m_j}^- v_j, \tag{3.3}$$

for some finite set L and $a_j \in \mathbb{C}^\times$.

Let $m_{j_0} = \min\{m_j \mid j \in L\}$ with respect to the order (3.2), and let $(\alpha_0, r_0) = \min\{(\alpha, r) \in \Delta^+ \times \mathbb{Z} \mid m_{j_0}(\alpha, r) > 0\}$ with respect to the order (3.1). In other words, m_{j_0} is the smallest element of M such that $x_{m_{j_0}}^-$ is a nontrivial summand in the expression (3.3), and $(x_{\alpha_0, r_0}^-)^{m_{j_0}(\alpha_0, r_0)}$ is its right most factor.

Let m' be the element of M such that $m'(\alpha, r) = m_{j_0}(\alpha, r)$ if $(\alpha, r) \neq (\alpha_0, r_0)$ and $m'(\alpha_0, r_0) = m_{j_0}(\alpha_0, r_0) - 1$. Let $\ell \in \mathbb{Z}^\times$ be such that neither $\phi_{i, \ell+r_0}$ or $\phi_{i, -\ell+r_0}$ is a factor in any v_j for all $j \in L$ and $i \in I$. Then, since $V \cong V(\mu, a, F)$, by Lemma 2.3 (see also Remark 2.6), we may choose the sign of ℓ so that $\phi_{i, \ell+r_0} v_{j_0} \neq 0$, and hence that $[x_{\alpha_0, \ell}^+, x_{\alpha_0, r_0}^-] v_{j_0} = H_{\alpha_0, \ell+r_0} v_{j_0} \neq 0$. Since $x_{\alpha_0, \ell}^+ v_j = 0$ for all $j \in J$, we obtain

$$x_{\alpha_0, \ell}^+ v = \sum_{j \in L} a_j [x_{\alpha_0, \ell}^+, x_{m_j}^-] v_j.$$

Now, we claim that there is only one occurrence of $x_{m'}^- h_{\alpha_0, \ell+r_0} v_{j_0}$ in the basis expansion of $x_{\alpha_0, \ell}^+ v$.

It follows from the order we fixed on the monomials of $U(\widehat{\mathfrak{g}})$ that the only way to have a nonzero bracket of elements of $\widehat{\mathfrak{g}}$ in the Leibniz expansion of $[x_{\alpha_0, \ell}^+, x_{m_j}^-]$ is if $x_{m_j}^-$ has factors of the form: (1) $x_{\alpha_0, k}^-$; (2) $x_{\alpha_0 - \beta, k}^-$ for $\beta < \alpha_0$; or (3) $x_{\alpha_0 + \beta, k}^-$ for $\alpha_0 < \alpha_0 + \beta$. For each factor in case (3) we obtain an element in $U(\mathcal{U}^-)$, and hence with no factor in \mathcal{H} . In the case (2), for each factor $x_{\alpha_0 - \beta, k}^-$ with $\beta < \alpha_0$ we get a factor of the form $x_{\beta, k+\ell}^+$. Since α_0 is minimal and $\beta < \alpha_0$, in the process of commuting $x_{\beta, k+\ell}^+$ it with the remaining factors of $x_{m_j}^-$ we can only produce elements of the form $x_{\gamma, p}^+$ with $\gamma < \alpha_0$ or elements in \mathcal{U}^- , and again we get elements with no factor in \mathcal{H} .

Finally, writing $x_{m_j}^- = u'_{m_j} u_{m_j}$ where $u'_{m_j} \in U(\mathcal{U}^-)$ does not have any factor of the form $x_{\alpha_0, *}$, we conclude that we just have to look at $u'_{m_j} [x_{\alpha_0, \ell}^+, u_{m_j}]$. Write $u_{m_j} = (x_{\alpha_0, r_{j_1}}^-)^{t_{j_1}} \cdots (x_{\alpha_0, r_{j_k}}^-)^{t_{j_k}}$, let $t = t_{j_1} + \cdots + t_{j_k}$, and notice that

$$\begin{aligned} u'_{m_j} [x_{\alpha_0, \ell}^+, u_{m_j}] &= u'_{m_j} \sum_i^t x_{\alpha_0, r_{j_1}}^- \cdots h_{\alpha_0, \ell+r_{j_i}} \cdots x_{\alpha_0, r_{j_k}}^- \\ &= u'_{m_j} \sum_i^t x_{\alpha_0, r_{j_1}}^- \cdots [h_{\alpha_0, \ell+r_{j_i}}, x_{\alpha_0, r_{j_i}}^- \cdots x_{\alpha_0, r_{j_k}}^-] \end{aligned}$$

$$\begin{aligned}
 &+ (x_{\alpha_0, r_{j_1}}^-)^{t_{j_1}} \cdots (x_{\alpha_0, r_{j_i}}^-)^{t_{j_i}-1} \cdots (x_{\alpha_0, r_{j_k}}^-)^{t_{j_k}} h_{\alpha_0, \ell+r_{j_i}} \\
 &= u'_{m_j} \sum_i^t (x_{\alpha_0, r_{j_1}}^-)^{t_{j_1}} \cdots (x_{\alpha_0, r_{j_i}}^-)^{t_{j_i}-1} \cdots (x_{\alpha_0, r_{j_k}}^-)^{t_{j_k}} h_{\alpha_0, \ell+r_{j_i}} \pmod{U(\mathcal{U}^-)}.
 \end{aligned}$$

Hence, the only way to have

$$u'_{m_j} (x_{\alpha_0, r_{j_1}}^-)^{t_{j_1}} \cdots (x_{\alpha_0, r_{j_i}}^-)^{t_{j_i}-1} \cdots (x_{\alpha_0, r_{j_k}}^-)^{t_{j_k}} h_{\alpha_0, \ell+r_{j_i}} = x_m^- h_{\alpha_0, \ell+r_0}$$

is if

$$\begin{aligned}
 u'_{m_j} &= u'_{m_{j_0}}, \\
 (x_{\alpha_0, r_{j_1}}^-)^{t_{j_1}} \cdots (x_{\alpha_0, r_{j_i}}^-)^{t_{j_i}-1} \cdots (x_{\alpha_0, r_{j_k}}^-)^{t_{j_k}} &= (x_{\alpha_0, r_{j_0}}^-)^{t_{j_0}} \cdots (x_{\alpha_0, r_0}^-)^{m_{j_0}(\alpha_0, r_0)-1},
 \end{aligned}$$

and $h_{\alpha_0, \ell+r_{j_i}} = h_{\alpha_0, \ell+r_0}$. But this implies that only one occurrence of $x_m^- h_{\alpha_0, \ell+r_0} v_{j_0}$ in the basis expansion of $x_{\alpha_0, \ell}^+ v$ comes from $[x_{\alpha_0, \ell}^+, x_{m_{j_0}}^-] v_{j_0}$. This shows that $x_{\alpha_0+\ell\delta}^+ v \neq 0$ and since $\text{ht}(x_{\alpha_0, \ell}^+ v) = \text{ht}(v) - \text{ht}(\alpha_0) < \text{ht}(v)$, the statement follows. \square

The next result generalizes the case $X = \emptyset$ of [2, Theorem 5.9] to the case where we let V be an arbitrary irreducible diagonal module. In the non-super case this was shown in [1].

Theorem 3.3. *Let $\mu \in \mathbb{C}^{I \times \mathbb{Z}_{>0}}$, $\lambda \in \mathfrak{h}^*$ with $\lambda(c) = a \neq 0$, and $F \subseteq F_{\mu, a}$. Let V be an \mathcal{H} -module of the form $V(\mu, a, F)$ or $L(\mu, a)$. For any nonzero submodule $N \subseteq M(\lambda, V)$, we have $N \cap (1 \otimes V) \neq 0$. In particular, $M(\lambda, V)$ is irreducible if and only if V is irreducible.*

Proof. Let $0 \neq v \in N$. If $v \in 1 \otimes V$, we are done. Suppose $v \notin 1 \otimes V$. Then $\text{ht}(v) > 0$, and Lemma 3.2 allows us to use induction on $\text{ht}(v)$ to prove the statement. \square

3.2. Kac modules

Recall that if \mathfrak{g} is one of Lie superalgebras $\mathfrak{gl}(m, n)$, $\mathfrak{sl}(m, n)$, $\mathfrak{psl}(n, n)$ or $\mathfrak{osp}(2, 2m)$ for $m > 1$, then we have a \mathbb{Z} -grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which is consistent with the \mathbb{Z}_2 -grading of \mathfrak{g} . Such decomposition induces a similar decomposition of the affine Kac-Moody Lie superalgebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_{-1} \oplus \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_1$. Set $\widehat{\mathfrak{g}}_+ = \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_1$. It is well known that there is a distinguished set of simple roots Σ for which $\mathfrak{n}^\pm = \mathfrak{n}_0^\pm \oplus \mathfrak{g}_{\pm 1}$. Fix such Σ and consider the corresponding parabolic subalgebra $\mathcal{P}_\Sigma = \mathfrak{h} \oplus \mathcal{H} \oplus \mathfrak{n}^+ \otimes \mathbb{C}[t, t^{-1}]$. We have that

$$\widehat{\mathfrak{g}}_+ = \mathfrak{n}_0^- \otimes \mathbb{C}[t, t^{-1}] \oplus \mathcal{P}_\Sigma \quad \text{and} \quad \widehat{\mathfrak{g}} \supset \widehat{\mathfrak{g}}_+ \supset \mathcal{P}_\Sigma. \tag{3.4}$$

For any $\widehat{\mathfrak{g}}_0$ -module L we can now define the *Kac module*

$$K(L) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_+)} L,$$

where we let $\widehat{\mathfrak{g}}_1$ act trivially on L .

Notice that the Heisenberg algebra \mathcal{H} coincides with the imaginary subalgebra of $\widehat{\mathfrak{g}}_0$. Thus, for any \mathcal{H} -module V and $\lambda \in \mathfrak{h}^*$ we can consider the $\widehat{\mathfrak{g}}_0$ -module

$$M_0(\lambda, V) = \text{Ind}(\mathcal{P}_0, U(\widehat{\mathfrak{g}}_0); V) = U(\widehat{\mathfrak{g}}_0) \otimes_{U(\mathcal{P}_0)} V,$$

where $\mathcal{P}_0 = \mathcal{P} \cap \widehat{\mathfrak{g}}_0 = \mathfrak{h} \oplus \mathcal{H} \oplus \mathfrak{n}_0^+ \otimes \mathbb{C}[t, t^{-1}]$. We also set $M_\Sigma(\lambda, V) = \text{Ind}(\mathcal{P}_\Sigma, U(\widehat{\mathfrak{g}}); V)$.

Theorem 3.4. *The following statements hold:*

- (i) *For any \mathcal{H} -module V , we have an isomorphism of $\widehat{\mathfrak{g}}$ -modules $M_\Sigma(\lambda, V) \cong K(M_0(\lambda, V))$.*
- (ii) *Let $\mu \in \mathbb{C}^{I \times \mathbb{Z}_{>0}}$, $\lambda \in \mathfrak{h}^*$ with $\lambda(c) = a \neq 0$, and $F \subseteq \mathbb{F}_{\mu,a}$. Let V be an \mathcal{H} -module of the form $V(\mu, a, F)$ or $L(\mu, a)$. Then the Kac module $K(M_0(\lambda, V))$ is irreducible if and only if $M_0(\lambda, V)$ is irreducible.*

Proof. Part (i): The inclusions (3.4) imply that the following isomorphism of $\widehat{\mathfrak{g}}$ -modules holds:

$$M_\Sigma(\lambda, V) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathcal{P}_\Sigma)} V \cong U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_+)} (U(\widehat{\mathfrak{g}}_+) \otimes_{U(\mathcal{P}_\Sigma)} V).$$

We claim now that the $\widehat{\mathfrak{g}}_+$ -modules $U(\widehat{\mathfrak{g}}_+) \otimes_{U(\mathcal{P}_\Sigma)} V$ and $M_0(\lambda, V)$ are isomorphic. It follows from their definition that both modules are isomorphic as $\widehat{\mathfrak{g}}_0$ -modules. Indeed, both modules are isomorphic to $U(\mathfrak{n}_0^- \otimes \mathbb{C}[t, t^{-1}]) \otimes_{\mathbb{C}} V$ as vector spaces, and the $\widehat{\mathfrak{g}}_0$ -action on them given by the left multiplication of $U(\widehat{\mathfrak{g}}_0)$. Now the claim follows because $\widehat{\mathfrak{g}}_1$ acts trivially on both modules. Hence, we have isomorphisms of $\widehat{\mathfrak{g}}$ -modules

$$\begin{aligned} M_\Sigma(\lambda, V) &\cong U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_+)} (U(\widehat{\mathfrak{g}}_+) \otimes_{U(\mathcal{P}_\Sigma)} V) \\ &\cong U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_+)} M_0(\lambda, V) \\ &= K(M_0(\lambda, V)). \end{aligned}$$

Part (ii): Note that $M_0(\lambda, V)$ is irreducible if and only if V is irreducible by Theorem 3.3. The statement follows from part (i) and Theorem 3.3. \square

4. Generalized imaginary Verma modules for $U_q \widehat{\mathfrak{sl}}(m|n)$

Throughout this section we assume that \mathfrak{g} is a Lie superalgebra of type $\mathfrak{sl}(m, n)$. Let $N = n + m$ and recall that for any given N -tuple $\mathbf{s} := (s_1, \dots, s_N) \in \{\pm 1\}^N$ we have a corresponding set of simple roots $\Sigma_{\mathbf{s}}$ and its corresponding Cartan matrix $A^{\mathbf{s}}$. From now on we fix \mathbf{s} and we set $\Sigma = \Sigma_{\mathbf{s}}$ and $A = A^{\mathbf{s}}$. Set also $Q = \mathbb{Z}\Sigma$ the root lattice corresponding to Σ . Our goal is to define analogs of generalized imaginary Verma modules for the quantum superalgebra $U_q(\widehat{\mathfrak{g}})$ and to prove that such modules are quantum deformations of generalized imaginary Verma modules over $\widehat{\mathfrak{g}}$.

4.1. $U_q(\widehat{\mathfrak{g}})$ and its \mathbb{A} -form

Let q be an indeterminate. In the *new Drinfeld realization*, the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$ is the associative superalgebra over $\mathbb{C}(q)$ generated by vectors $X_{i,n}^\pm, H_{i,r}, K_i^{\pm 1}, q^{\pm c}, q^{\pm d}, i \in I := \{1, \dots, |\Sigma|\}, n \in \mathbb{Z}, r \in \mathbb{Z}^\times$, where their parities are given by $|X_{i,r}^\pm| = |i| = (1 - s_i s_{i+1})/2$, and all remaining generators are even. The defining relations of $U_q(\widehat{\mathfrak{g}})$ are given as follows:

$$\begin{aligned}
 & q^{\pm c} \text{ is central, } K_i K_j = K_j K_i, \quad K_i X_j^\pm(z) K_i^{-1} = q^{\pm A_{i,j}} X_j^\pm(z), \\
 & q^d X_{i,r}^\pm q^{-d} = q^r X_{i,r}^\pm, \quad q^d H_{i,r} q^{-d} = q^r H_{i,r}, \quad q^d K_i q^{-d} = K_i, \\
 & [H_{i,r}, H_{j,s}] = \delta_{r+s,0} \frac{[r A_{i,j}]}{r} \frac{q^{rc} - q^{-rc}}{q - q^{-1}}, \\
 & [H_{i,r}, X_j^\pm(z)] = \pm \frac{[r A_{i,j}]}{r} q^{-c(r \pm |r|)/2} z^r X_j^\pm(z), \\
 & [X_i^+(z), X_j^-(w)] = \frac{\delta_{i,j}}{q - q^{-1}} \left(\delta(q^c \frac{w}{z}) K_i^+(w) - \delta(q^c \frac{z}{w}) K_i^-(z) \right), \\
 & (z - q^{\pm A_{i,j}} w) X_i^\pm(z) X_j^\pm(w) + (-1)^{|i||j|} (w - q^{\pm A_{i,j}} z) X_j^\pm(w) X_i^\pm(z) = 0 \quad (A_{i,j} \neq 0), \\
 & [X_i^\pm(z), X_j^\pm(w)] = 0 \quad (A_{i,j} = 0), \\
 & \text{Sym}_{z_1, z_2} \llbracket X_i^\pm(z_1), [X_i^\pm(z_2), X_{i \pm 1}^\pm(w)] \rrbracket = 0 \quad (A_{i,i} \neq 0, i \pm 1 \in I), \\
 & \text{Sym}_{z_1, z_2} \llbracket X_i^\pm(z_1), [X_{i+1}^\pm(y), [X_i^\pm(z_2), X_{i-1}^\pm(w)]] \rrbracket = 0 \quad (A_{i,i} = 0, i \pm 1 \in I),
 \end{aligned}$$

where z, w, z_1, z_2 are formal commutative variables,

$$\begin{aligned}
 X_i^\pm(z) & := \sum_{k \in \mathbb{Z}} X_{i,k}^\pm z^{-k}, & \delta(z) & := \sum_{k \in \mathbb{Z}} z^k \\
 K_i^\pm(z) & := \sum_{r \in \mathbb{Z}_{\geq 0}} K_{i, \pm r}^\pm z^{\mp r} = K_i^{\pm 1} \exp \left(\pm (q - q^{-1}) \sum_{r > 0} H_{i, \pm r} z^{\mp r} \right),
 \end{aligned}$$

and

$$\llbracket X, Y \rrbracket = [X, Y]_{q^{-\beta(h_\gamma)}} := XY - (-1)^{|X||Y|} q^{-\beta(h_\gamma)} YX$$

if X and Y have weights $\beta, \gamma \in Q$, respectively (i.e. $K_i X K_i^{-1} = q^{\beta(h_i)} X$ and $K_i Y K_i^{-1} = q^{\gamma(h_i)} Y$).

Notice that the defining relations depend on the choice of Σ , however, it is known that the superalgebra $U_q(\widehat{\mathfrak{g}})$ is independent of such choice. Finally, we define the following Lie subalgebras of $U_q(\widehat{\mathfrak{g}})$:

- \mathcal{H}_q^\pm the subalgebra generated by $H_{i, \pm r}, i \in I, r \in \mathbb{Z}_{>0}$;
- \mathcal{H}_q the subalgebra generated by \mathcal{H}_q^\pm, q^d and q^c ;

- $U_q(\widehat{\mathfrak{h}})$ the subalgebra generated by $K_i, i \in I, q^c$ and q^d ;
- \mathcal{U}_q^0 the subalgebra generated by \mathcal{H}_q and $U_q(\widehat{\mathfrak{h}})$;
- \mathcal{U}_q^\pm the subalgebra generated by $X_{i,r}^\pm, i \in I, r \in \mathbb{Z}$;
- \mathcal{P}_q the subalgebra generated by \mathcal{U}_q^0 and \mathcal{U}_q^+ ;
- \mathcal{B}_q the subalgebra generated by $U_q(\widehat{\mathfrak{h}}), \mathcal{H}_q^+$ and \mathcal{U}_q^+ ;
- $\mathcal{N}_q^{\pm\varphi}$ the subalgebra generated by $\mathcal{H}_q^{\pm\varphi}$ and \mathcal{U}_q^+ .

4.2. The PBWD basis

We recall the PBW basis constructed in [13]. For this we consider the total orders on Δ and on $\Delta^+ \times \mathbb{Z}$ as in Section 3.1. For each $(\alpha, r) \in \Delta^+ \times \mathbb{Z}$, write $\alpha = \alpha_{i_1} + \dots + \alpha_{i_p}, \alpha_{i_j} \in \Sigma$, and fix:

- (i) a decomposition $r = r_1 + \dots + r_p, r_i \in \mathbb{Z}$;
- (ii) a sequence $(q_1, \dots, q_{p-1}) \in \{q, q^{-1}\}^{p-1}$.

Now, each $(\alpha, r) \in \Delta^+ \times \mathbb{Z}$ defines a vector $X_{\pm\alpha,r} \in \mathcal{U}_q^\pm$ as

$$X_{\pm\alpha,r} := [\dots [[X_{i_1,r_1}^\pm, X_{i_2,r_2}^\pm]_{q_1}, X_{\alpha_{i_3},r_3}^\pm]_{q_3}, \dots, X_{i_p,r_p}^\pm]_{q_{p-1}}.$$

We have that $K_i X_{\pm\alpha,r} K_i^{-1} = q^{\pm\alpha(h_i)} X_{\pm\alpha,r}$, that is, $X_{\pm\alpha,r}$ is a root vector whose associated root is $\alpha + r\delta \in \Delta^+ \times \mathbb{Z}$.

As in Section 3.1, we let \mathbb{M} be the set of all functions $\mathfrak{m} : \Delta^+ \times \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ with finite support and such that $\mathfrak{m}(\alpha, k) \leq 1$ if $|\alpha| = 1$, and, for each $\mathfrak{m} \in \mathbb{M}$ we define monomials

$$X_{\mathfrak{m}}^+ := \prod_{(\alpha,r) \in \Delta^+ \times \mathbb{Z}}^{\rightarrow} (X_{\alpha,r})^{\mathfrak{m}(\alpha,r)}, \quad X_{\mathfrak{m}}^- := \prod_{(\alpha,r) \in \Delta^+ \times \mathbb{Z}}^{\leftarrow} (X_{-\alpha,r})^{\mathfrak{m}(\alpha,r)}.$$

These monomials are called ordered PBWD monomials of \mathcal{U}_q^+ and \mathcal{U}_q^- , respectively. As before, the arrow \rightarrow over the product indicates that the product is written in increasing order from left to right with respect to the order on $\Delta^+ \times \mathbb{Z}$, and the opposite for \leftarrow .

Theorem 4.1. [13, Theorem 5.7] *The set of ordered monomials $\{X_{\mathfrak{m}}^\pm \mid \mathfrak{m} \in \mathbb{M}\}$ forms a linear basis of \mathcal{U}_q^\pm .*

Finally, let \mathbb{H} be the set of all functions $\mathfrak{h} : I \times \mathbb{Z} \rightarrow \mathbb{Z}$ with finite support and such that $\mathfrak{h}(i, r) \geq 0$ if $r \neq 0$, and define the monomials

$$K_{\mathfrak{h}}^\pm := \prod_{(i,r) \in I \times \mathbb{Z}_{>0}} (K_{i,\pm r}^\pm)^{\mathfrak{h}(i,\pm r)}, \quad K_{\mathfrak{h}}^0 := \prod_{i \in I} (K_i)^{\mathfrak{h}(i,0)}, \quad K_{\mathfrak{h}} := K_{\mathfrak{h}}^- K_{\mathfrak{h}}^0 K_{\mathfrak{h}}^+.$$

Theorem 4.2. [13, Remark 5.10] *The set of monomials*

$$\{X_{\mathfrak{m}}^- K_{\mathfrak{h}} q^{rc} q^{r'd} X_{\mathfrak{m}'}^+ \mid \mathfrak{m}, \mathfrak{m}' \in \mathbb{M}, \mathfrak{h} \in \mathbb{H}, r, r' \in \mathbb{Z}\}$$

is a linear basis of $U_q(\widehat{\mathfrak{g}})$. In particular, we have a triangular decomposition $U_q(\widehat{\mathfrak{g}}) \cong \mathcal{U}_q^- \otimes \mathcal{U}_q^0 \otimes \mathcal{U}_q^+$.

4.3. Quantum induced modules

Set $P_{\mathbb{Z}} := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z} \ \forall i \in I\}$. Let $V_q \in \mathcal{K}_q$ and $\lambda \in P_{\mathbb{Z}}$. Extend V_q to a \mathcal{P}_q -module by setting

$$X_{i,n}^+ V_q = 0, \quad K_i^{\pm} v = q^{\pm \lambda(h_i)} v$$

for any $v \in V_q$. Define the induced $U_q(\widehat{\mathfrak{g}})$ -module $M_q(\lambda, V_q)$ as

$$M_q(\lambda, V_q) := \text{Ind}(\mathcal{P}_q, U_q(\widehat{\mathfrak{g}}); V_q).$$

If V_q is an irreducible \mathcal{H}_q -module, then $M_q(\lambda, V_q)$ is called a *quantum generalized imaginary Verma module* associated with \mathcal{P}_q , V_q and λ . If V_q is a φ -Verma module of \mathcal{H}_q , then $M_q^\varphi(\lambda) := M_q(\lambda, V_q)$ is called a *quantum imaginary φ -Verma module* of $U_q(\widehat{\mathfrak{g}})$.

We have the following basic properties of $M_q(\lambda, V_q)$:

Proposition 4.3. *Let V_q be an irreducible \mathcal{H}_q -module in \mathcal{K}_q with a $\mathbb{C}(q)$ -linear basis $\{v_j\}_{j \in J}$, and let $\lambda \in \widehat{\mathfrak{h}}^*$ with $\lambda(c) \neq 0$. Then,*

- (i) $M_q(\lambda, V_q)$ is a free \mathcal{U}_q^- -module generated by $\{v_j\}_{j \in J}$;
- (ii) $\dim M_q(\lambda, V_q)_\mu = \infty$ for any $\mu = \lambda - \beta + n\delta$, $\beta \in Q^+ \setminus \{0\}$, $n \in \mathbb{Z}$;
- (iii) $\dim M_q(\lambda, V_q)_\mu < \infty$ if and only if V_q is a φ -Verma module of \mathcal{H}_q with φ constant and $\mu = \lambda - \varphi(n)n\delta$, $n \in \mathbb{Z}_{\geq 0}$;
- (iv) $M_q^\varphi(\lambda)$ is a free $\mathcal{N}_q^{-\varphi}$ -module generated by $1 \otimes v_\varphi$.

Proof. The statements (i) and (ii) follow from the triangular decomposition $U_q(\widehat{\mathfrak{g}}) = \mathcal{U}_q^- \otimes \mathcal{U}_q^0 \otimes \mathcal{U}_q^+$ and the PBW basis of \mathcal{U}_q^- . Item (iv) is similar, noting that we also have the decomposition

$$U_q(\widehat{\mathfrak{g}}) = \mathcal{N}_q^{-\varphi} \otimes U_q(\widehat{\mathfrak{h}}) \otimes \mathcal{N}_q^\varphi.$$

To prove item (iii), notice that if $\dim M_q(\lambda, V_q)_\mu < \infty$, then $\mu = \lambda + n\delta$ for some $n \in \mathbb{Z}$, and $M_q(\lambda, V_q)_\mu = 1 \otimes V_{q,n}$. But $\dim V_{q,n} < \infty$ if and only if V_q is a φ -Verma module of \mathcal{H}_q with φ constant. \square

4.4. \mathbb{A} -forms

Recall that $\mathbb{A} = \{\frac{f(q)}{g(q)} \in \mathbb{C}(q) \mid g(1) \neq 0\}$. For $i \in I$, $k, l, n \in \mathbb{Z}$ with $n > 0$, and $Y \in U_q(\mathfrak{h})$, we define

$$\begin{aligned} \left[\begin{matrix} K_i; k, l \\ n \end{matrix} \right] &= \prod_{r=1}^n \frac{q^{kc} K_{i,k+l}^+ - q^{lc} K_{i,k+l}^-}{q^r - q^{-r}}, \\ \left[\begin{matrix} Y; k \\ n \end{matrix} \right] &= \prod_{r=1}^n \frac{Y q^{k-r+1} - Y^{-1} q^{r-k-1}}{q^r - q^{-r}}. \end{aligned}$$

Let $U_{\mathbb{A}}(\widehat{\mathfrak{g}})$ denote the \mathbb{A} -subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $X_{i,r}^{\pm}, H_{i,\pm n}, K_i^{\pm 1}, q^{\pm c}, q^{\pm d}, \left[\begin{matrix} K_i; k \\ n \end{matrix} \right], \left[\begin{matrix} q^c; k \\ n \end{matrix} \right], \left[\begin{matrix} q^d; k \\ n \end{matrix} \right], \left[\begin{matrix} K_i; k, l \\ n \end{matrix} \right], i \in I, k, l, n, r \in \mathbb{Z}, n > 0$.

By straightforward calculations, we have the following proposition.

Proposition 4.4. *The generators of $U_{\mathbb{A}}(\widehat{\mathfrak{g}})$ satisfy the following relations.*

- (i) $q^{\pm c}$ is central, $K_i K_j = K_j K_i, K_i x_j^{\pm}(z) K_i^{-1} = q^{\pm A_{i,j}} X_j^{\pm}(z);$
- (ii) $q^d X_{i,r}^{\pm} q^{-d} = q^r X_{i,r}^{\pm};$
- (iii) $\left[\begin{matrix} K_j; k \\ n \end{matrix} \right] X_{i,r}^{\pm} = X_{i,r}^{\pm} \left[\begin{matrix} K_j; k \pm A_{i,j} \\ n \end{matrix} \right];$
- (iv) $\left[\begin{matrix} q^d; k \\ n \end{matrix} \right] X_{i,r}^{\pm} = X_{i,r}^{\pm} \left[\begin{matrix} q^d; k + r \\ n \end{matrix} \right];$
- (v) $\left[\begin{matrix} q^d; k \\ n \end{matrix} \right] H_{i,r}^{\pm} = H_{i,r}^{\pm} \left[\begin{matrix} q^d; k + r \\ n \end{matrix} \right];$
- (vi) $[H_{i,k}, H_{j,l}] = \delta_{k-l,0} \frac{[kA_{i,j}]}{k} \left[\begin{matrix} q^c; k \\ 1 \end{matrix} \right];$
- (vii) $[X_{i,r}^+, X_{j,s}^-] = \delta_{i,j} \left[\begin{matrix} K_i; r, s \\ 1 \end{matrix} \right].$
- (viii) $[H_{i,r}, X_j^{\pm}(z)] = \pm \frac{[rA_{i,j}]}{r} q^{-c(r \pm |r|)/2} z^r X_j^{\pm}(z),$

As a consequence of Proposition 4.4 we have the following corollary. The proof in the super case is identical to that in the even case given in [3].

Corollary 4.5. *Let $U_{\mathbb{A}}^{\pm} = U_{\mathbb{A}}(\widehat{\mathfrak{g}}) \cap \mathcal{U}_q^{\pm}$ and $U_{\mathbb{A}}^0 = U_{\mathbb{A}}(\widehat{\mathfrak{g}}) \cap \mathcal{U}_q^0$. Then, the subalgebra $U_{\mathbb{A}}(\widehat{\mathfrak{g}})$ has the triangular decomposition*

$$U_{\mathbb{A}}(\widehat{\mathfrak{g}}) = U_{\mathbb{A}}^- \otimes U_{\mathbb{A}}^0 \otimes U_{\mathbb{A}}^+.$$

Note that the generators of $U_{\mathbb{A}}^{\pm}$ and U_q^{\pm} coincide, and all coefficients obtained through commuting the generators lie in \mathbb{A} . Therefore, the \mathbb{A} version of Theorem 4.1 holds, i.e., the PBWD monomials $X_m^{\pm}, m \in \mathbb{M}$, form a linear basis of $U_{\mathbb{A}}^{\pm}$.

Now we construct the \mathbb{A} -form of diagonal \mathcal{H}_q -modules.

Let $\mu \in \mathbb{C}(q)^{I \times \mathbb{Z}_{>0}}$ and

$$V_q \in \{V_q(\mu, a, F) \mid F \subseteq F_{\mu,a}\} \quad (\text{resp. } V \in \{V(\mu, a, F) \mid F \subseteq F_{\mu,a}\})$$

as defined in Section 2.3 (see also Remark 2.5). Recall that V_q is isomorphic to a quotient of \mathcal{H}_q by a left ideal $\mathcal{L}_{\mu,a}$ contained in the left ideal generated by the set $\{\phi_{i,k}\phi_{i,-k} - \mu_{i,k}, q^c - q^a, q^d \mid (i, k) \in I \times \mathbb{Z}_{>0}\} \cup \{(\phi_{i,-\varphi(i,k)k})^{\frac{\mu_{i,k}}{[ka]}} \mid \frac{\mu_{i,k}}{[ka]} > 0, \forall (i, k) \in F_{\mu,a}\} \cup \{(\phi_{i,-\varphi(i,k)k})^{1-\frac{\mu_{i,k}}{[ka]}} \mid \frac{\mu_{i,k}}{[ka]} \leq 0, \forall (i, k) \in F_{\mu,a}\}$. In fact, if $V_q = V_q(\mu, a, F_{\mu,a})$ then $\mathcal{L}_{\mu,a}$ equals this set.

Let $\mathcal{H}_{\mathbb{A}} = U_{\mathbb{A}}(\widehat{\mathfrak{g}}) \cap \mathcal{H}_q$. Since $\mu_{i,k} \in \mathbb{C}(q)$ for all $(i, k) \in I \times \mathbb{Z}_{>0}$, say $\mu_{i,k} = \frac{f_{i,k}(q)}{g_{i,k}(q)}$, we can choose generators of $\mathcal{L}_{\mu,a}$ in $\mathcal{H}_{\mathbb{A}}$ (just multiply each $\phi_{i,k}\phi_{i,-k} - \mu_{i,k}$ by $g_{i,k}(q)$). Let $\mathcal{L}_{\mu,a}^{\mathbb{A}}$ be the ideal of $\mathcal{H}_{\mathbb{A}}$ generated by these elements, and define

$$V_{\mathbb{A}} = \mathcal{H}_{\mathbb{A}} / \mathcal{L}_{\mu,a}^{\mathbb{A}}.$$

Remark 4.6. Usually the \mathbb{A} -form of a module V_q is constructed by fixing a suitable basis of V_q and taking the \mathbb{A} -form of V_q to be the \mathbb{A} -module generated by such basis. Then it is proved that such \mathbb{A} -module is invariant under the action of the algebra generators. We notice that this construction coincides with ours in the case that $\mu \in \mathbb{A}^{I \times \mathbb{Z}_{>0}}$, i.e., there is no $\mu_{i,k}$ with pole at $q = 1$. Hence, assuming that $\mu \in \mathbb{A}^{I \times \mathbb{Z}_{>0}}$, we have that the basis $\{v_j\}_{j \in J}$ of V_q constructed in Section 2.3 is also an \mathbb{A} -basis for $V_{\mathbb{A}}$. Moreover, we will see below that if $\mu_{i,k}$ has a pole at $q = 1$, then the classical limit of $V_{\mathbb{A}}$ will vanish.

The \mathbb{A} -form $M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})$ of $M_q(\lambda, V_q)$ is defined as the $U_{\mathbb{A}}$ -submodule of $M_q(\lambda, V_q)$ generated by $V_{\mathbb{A}}$. That is

$$M_{\mathbb{A}}(\lambda, V_{\mathbb{A}}) := \text{Ind}(\mathcal{P}_{\mathbb{A}}, U_{\mathbb{A}}; V_{\mathbb{A}}).$$

As a consequence of the PBW Theorem and the triangular decompositions of $U_{\mathbb{A}}$, we have the \mathbb{A} -form version of Proposition 4.3.

Lemma 4.7. *Let V_q be a \mathcal{H}_q -module as above and assume that there is no $\mu_{i,k}$ with pole at $q = 1$. Let $\{v_j\}_{j \in J}$ be an \mathbb{A} -basis of $V_{\mathbb{A}}$ as in Remark 4.6, and let $\lambda \in P$ with $\lambda(c) \neq 0$. Then the following statements hold:*

- (i) $M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})$ is a free $\mathcal{U}_{\mathbb{A}}^-$ -module generated by $\{1 \otimes v_j\}_{j \in J}$;
- (ii) $M_{\mathbb{A}}^{\varphi}(\lambda)$ is a free $\mathcal{N}_{\mathbb{A}}^{-\varphi}$ -module generated by $1 \otimes v_{\varphi}$.

Recall that $M_q(\lambda, V_q)$ has a basis $\{X_m^- v_j \mid m \in M, j \in J\}$. Thus, we have a $\mathbb{C}(q)$ -linear map

$$M_q(\lambda, V) \rightarrow \mathbb{C}(q) \otimes_{\mathbb{A}} M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})$$

given by $X_m^- v_j \mapsto 1 \otimes X_m^- v_j$, whose inverse is given by $f \otimes v \mapsto fv$ for all $f \in \mathbb{C}(q)$ and $v \in M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})$. Notice that this isomorphism preserves weight spaces. That is, we have the following lemma.

Lemma 4.8. *The following isomorphisms of $\mathbb{C}(q)$ -vector spaces hold:*

- (i) $M_q(\lambda, V_q) \cong \mathbb{C}(q) \otimes_{\mathbb{A}} M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})$;
- (ii) $M_q(\lambda, V_q)_{\mu} \cong \mathbb{C}(q) \otimes_{\mathbb{A}} M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})_{\mu}$, where $M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})_{\mu} = M_q(\lambda, V_q)_{\mu} \cap M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})$;
- (iii) $M_{\mathbb{A}}(\lambda, V_{\mathbb{A}}) = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})_{\mu}$.

4.5. *Classical limit*

Consider $U' = \mathbb{A}/\langle q - 1 \rangle \otimes_{\mathbb{A}} U_{\mathbb{A}}(\widehat{\mathfrak{g}})$ and $\overline{U} = U'/K'$, where K' is the ideal of U' generated by $\{K_i - 1, q^c - 1, q^d - 1\}$. Let \overline{X} denote the image in \overline{U} of an element $X \in U'$. It was proved in [14, Lemma 6.6.1] that $\overline{U} \cong U(\widehat{\mathfrak{g}})$. Moreover, this isomorphism maps the generator $\overline{X_{\alpha,k}}$ to $x_{\alpha,k}$ for all $\alpha \in \widehat{\Delta}$ and $k \in \mathbb{Z}$.

Let $M_{\mathbb{A}}$ be equal to $V_{\mathbb{A}}$ or $M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})$ as considered in the previous section. We define the classical limit of $M_{\mathbb{A}}$ to be

$$\overline{M}_{\mathbb{A}} = \mathbb{A}/\langle q - 1 \rangle \otimes_{\mathbb{A}} M_{\mathbb{A}}.$$

For all $m \in M_{\mathbb{A}}$, we will denote the vector $1 \otimes m \in \overline{M}_{\mathbb{A}}$ by \overline{m} . Since $\mathcal{H} \cong \mathbb{A}/\langle q - 1 \rangle \otimes_{\mathbb{A}} \mathcal{H}_{\mathbb{A}}$, we have that $\overline{V}_{\mathbb{A}}$ is a module over \mathcal{H} . Moreover, since $K'\overline{M}_{\mathbb{A}} = 0$ we also conclude that $\overline{M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})}$ is a module over $U(\widehat{\mathfrak{g}})$.

Remark 4.9. Assume that, for some $(i, k) \in I \times \mathbb{Z}_{>0}$, we have $\mu_{i,k} = \frac{f(q)}{(q-1)q(q)}$, that is, $\mu_{i,k}$ has a pole at $q = 1$. In this case $\overline{V}_{\mathbb{A}} = 0$, and hence $\overline{M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})} = 0$.

Recall that

$$V_q \in \{V_q(\boldsymbol{\mu}, a, F) \mid F \subseteq F_{\boldsymbol{\mu},a}\} \quad (\text{resp. } V \in \{V(\boldsymbol{\mu}, a, F) \mid F \subseteq F_{\boldsymbol{\mu},a}\}).$$

Remark 4.9 motivates the following definition.

Definition 4.10. We call V_q *limit faithful* if there is no $\mu_{i,k}$ with pole at $q = 1$.

Lemma 4.11. *Assume that V_q is limit faithful. Then we have isomorphisms: $\overline{V}_{\mathbb{A}} \cong V$ as \mathcal{H} -modules, and $\overline{M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})} \cong M(\lambda, V)$ as $\widehat{\mathfrak{g}}$ -modules.*

Proof. The former isomorphism follows from the fact that the linear isomorphism that maps $\overline{v_j}$ to v_j is a homomorphism of \mathcal{H} -modules. For the latter isomorphism, we similarly consider the linear isomorphism that maps $\overline{X_m^- v_j}$ to $x_m^- v_j$. The fact that this is a homomorphism of $\widehat{\mathfrak{g}}$ -modules follows from the fact that $\overline{U} \cong U(\widehat{\mathfrak{g}})$ and that such isomorphism

of algebras maps the generator $\overline{X_{i,k}^\pm}$ to $x_{i,k}^\pm$ for all $i \in I$ and $k \in \mathbb{Z}$. Thus, for any $\overline{X_m^-}$ of \overline{U} we have that the PBW expansion of $\overline{X_{i,k}^\pm} \overline{X_m^-}$ in \overline{U} is mapped to the PBW expansion of $x_{i,k}^\pm x_m^-$ in $U(\widehat{\mathfrak{g}})$. \square

Lemma 4.12. *Let V_q be limit faithful, and let $\lambda \in P_{\mathbb{Z}}$ with $\lambda(c) \neq 0$. Then, for any nonzero weight vector $v \in M_q(\lambda, V_q)$ with $\text{ht}(v) > 0$, there exists $X \in U_q(\widehat{\mathfrak{g}})$ such that $Xv \neq 0$ and $\text{ht}(Xv) < \text{ht}(v)$.*

Proof. Let $v \in M_q(\lambda, V)$ be a nonzero vector of weight μ such that $\text{ht}(v) > 0$. Thus, μ must be of the form $\mu = \lambda - \beta + n\delta$, for some $\beta \in Q^+ \setminus \{0\}$, $n \in \mathbb{Z}$. Write

$$v = \sum_{j \in L} a_j(q) X_{m_j}^- w_j$$

as in Lemma 3.2.

Without loss of generality we may assume that $a_j(q) \in \mathbb{A}$ for all $j \in L$ and $a_j(1) \neq 0$ for some $j \in L$. Hence $v \in M_{\mathbb{A}}(\lambda, V_{\mathbb{A}}) \setminus \{0\}$, and

$$\bar{v} = \sum_{j \in L} a_j(1) \overline{X_{m_j}^-} \bar{v}_j \neq 0$$

in $\overline{M_{\mathbb{A}}(\lambda, V_{\mathbb{A}})} \cong M(\lambda, V)$ by Lemma 4.11. By Lemma 3.2, there exists $x_{\alpha,k} \in \mathcal{U}^+$ such that $x_{\alpha,k} \bar{v} \neq 0$ and $\text{ht}(x_{\alpha,k} \bar{v}) < \text{ht}(\bar{v})$. This implies $X_{\alpha,k} v \neq 0$. Since $\text{ht}(X_{\alpha,k} v) < \text{ht}(v)$ the statement follows. \square

Theorem 4.13. *Let V_q be limit faithful, and let $\lambda \in P_{\mathbb{Z}}$ with $\lambda(c) \neq 0$. For any non-trivial submodule $W \subseteq M_q(\lambda, V_q)$, we have $W \cap (1 \otimes V_q) \neq 0$.*

Proof. Let $w \in W$ and assume w is a weight vector. If $\text{ht}(w) = 0$, then $w \in 1 \otimes V$. If $\text{ht}(w) > 0$, then by Lemma 4.12, there exists $X \in \mathcal{U}_q^+$ such that $Xw \neq 0$ and the statement follows by induction. \square

From Theorem 4.13 we immediately obtain the following irreducibility criterion for quantum generalized imaginary Verma modules.

Corollary 4.14. *Let V_q be limit faithful, and let $\lambda \in P_{\mathbb{Z}}$ with $\lambda(c) \neq 0$. Then $M_q(\lambda, V_q)$ is irreducible if and only if V_q is irreducible.*

In particular, we get the irreducibility criterion for quantum imaginary φ -Verma modules.

Corollary 4.15. *Let $\lambda \in P_{\mathbb{Z}}$ with $\lambda(c) \neq 0$ and $\varphi : I \times \mathbb{Z}_{>0} \rightarrow \{\pm\}$. Then the quantum imaginary φ -Verma module $M_q^\varphi(\lambda)$ is irreducible.*

Data availability

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